

A Random Point Field related to Bose-Einstein Condensation

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Abstract

The random point field which describes the position distribution of the system of ideal boson gas in a state of Bose-Einstein condensation is obtained through the thermodynamic limit. The resulting point field is given by convolution of two independent point fields: the so called boson process whose generating functional is represented by inverse of the Fredholm determinant for an operator related to the heat operator and the point field whose generating functional is represented by a resolvent of the operator. The construction of the latter point field in an abstract formulation is also given.

1 Introduction

In the previous paper [TI], which we will refer as I, the authors gave a method which derives typical kinds of random point fields, the boson point process and the fermion point process on \mathbb{R}^d , through the thermodynamic limit from random point fields of fixed finite numbers of points in bounded boxes in \mathbb{R}^d . The purpose of the paper is to give the random point field which describes the position distribution of the system of ideal boson gas in a state of Bose-Einstein condensation [BEC] as an extension of I.

Let us consider the system of N free bosons in a box of finite volume V in \mathbb{R}^d and the quantum statistical mechanical state for the system of a finite temperature. Regarding the square of the absolute value of the wave function as the distribution function of the positions of N particles together with thermal average, we obtain a random point field of N points in the box. In I, the thermodynamic limit, $N, V \rightarrow \infty$ and $N/V \rightarrow \rho$, of the system for small ρ as well as the system of fermions for every positive ρ are taken to get the boson as well as the fermion point processes on \mathbb{R}^d in a simple and straightforward way. As applications of the approach, the system of para-particles and the system of composite particles are studied. The argument is based on the unified formulation of boson/fermion processes of [ST]. For general references of this field, see e.g. [So] and references cited there in.

In this paper, we study the case of large ρ (corresponding to BEC) which needs technically elaborate analysis for the largest eigenvalue $\tilde{g}_0(L)$ of the deformed heat operator \tilde{G}_L in the box of size L and the saddle points z_0 and \tilde{z}_0 for complex integrals related to generalized Vere-Jones' formula [V, ST] more than those in I. As the result of the thermodynamic limit, we get a random point field on \mathbb{R}^d which are given by convolution of two independent point fields: 1. the boson process whose generating functional is represented by inverse of the Fredholm determinant for an operator related to the heat operator; 2. the point field whose generating functional is represented by a resolvent of the operator.

The paper organized as follows: In §2 the construction of the point field which appears in the resulting point fields as the second independent component (see above). The construction is made in a general framework of random point fields similar to [ST], i.e., on the locally compact space of second countability. §3 devoted to the analysis of the thermodynamic limit in \mathbb{R}^d .

2 Abstract formulation of the random point field

Let R be a locally compact Hausdorff space with countable basis and λ a positive Radon measure on R . We regard λ as a measure on the Borel σ -algebra $\mathcal{B}(R)$ which assigns finite values for compact sets. Relatively compact subsets of R will be called bounded. On $L^2(R; \lambda)$, we consider a (possibly unbounded) non-negative self-adjoint operator K which satisfies:

Condition K

- (i) [*locally boundedness*] For any bounded measurable subset Λ of R , the operator $K^{1/2}\chi_\Lambda$ is bounded, where χ_Λ denotes the operator multiplying the indicator function χ_Λ .
- (ii) $G = K(1 + K)^{-1}$ has a non-negative integral kernel $G(x, y)$ which satisfies

$$\int_R G(x, y)\lambda(dy) \leq 1 \quad \lambda - a.e. x \in R. \quad (2.1)$$

For a measurable function $f : R \rightarrow [0, \infty)$ with compact support and a bounded measurable set Λ satisfying $\Lambda \supset \text{supp } f$, we have $K^{1/2}\sqrt{1 - e^{-f}} = K^{1/2}\chi_\Lambda\sqrt{1 - e^{-f}}$ and hence that

$$K_f = (K^{1/2}\sqrt{1 - e^{-f}})^* K^{1/2}\sqrt{1 - e^{-f}} \quad (2.2)$$

is a bounded non-negative self-adjoint operator. Here we regard $\sqrt{1 - e^{-f}}$ the multiplication operator of the function expressed by the same symbol. $Q(R)$ denotes the Polish space of all the locally finite non-negative integer valued Borel measures on R .

Theorem 2.1 For R, λ and K which satisfy the above conditions and $\rho > 0$, there exists a unique Borel probability measure $\mu_{K,\rho}$ on $Q(R)$ such that

$$\int_{Q(R)} e^{-\langle f, \xi \rangle} d\mu_{K,\rho}(d\xi) = \exp \left(-\rho(\sqrt{1 - e^{-f}}, [1 + K_f]^{-1} \sqrt{1 - e^{-f}}) \right) \quad (2.3)$$

holds for any non-negative measurable function f on R with compact support, where (\cdot, \cdot) denotes the inner product of $L^2(R; \lambda)$.

Let us begin with some remarks before proving the theorem. It follows that G is self-adjoint and $0 \leq G \leq 1$, where 1 denotes the identity operator on $L^2(R; \lambda)$. Without loss of generality, we may assume that the $\mathcal{B}(R^2)$ -measurable function $G(x, y)$ satisfies

$$\forall x, y \in R: \quad G(x, y) \geq 0, \quad G(x, y) = G(y, x)$$

and

$$\forall x \in R: \quad \int_R G(x, y) \lambda(dy) \leq 1.$$

Let us define the functions $G^n(x, y)$ inductively as

$$G^1(x, y) = G(x, y), \quad \text{and} \quad G^{n+1}(x, y) = \int_R G^n(x, z) G(z, y) \lambda(dz) \quad \text{for } n \in \mathbb{N}.$$

Then we have

$$\forall x, y \in R, \forall n \in \mathbb{N}: \quad G^n(x, y) \geq 0, \quad G^n(x, y) = G^n(y, x)$$

and

$$\forall x \in R, \forall n \in \mathbb{N}: \quad \int_R G^n(x, y) \lambda(dy) \leq 1.$$

It is obvious that $G^n(x, y)$ is the integral kernel of the operator G^n for any $n \in \mathbb{N}$.

Put

$$K_n = \sum_{k=1}^n G^k \quad \text{and} \quad K_n(x, y) = \sum_{k=1}^n G^k(x, y).$$

Then K_n is the bounded non-negative self-adjoint operator which has non-negative integral kernel $K_n(x, y)$. The function

$$K(x, y) = \lim_{n \rightarrow \infty} K_n(x, y) = \sum_{k=1}^{\infty} G^k(x, y) \quad (2.4)$$

is well defined, if we admit infinity as its value.

Here we recall the following preliminary facts from functional analysis.

Lemma 2.2 (i) Let \mathcal{H} be a Hilbert space, $\mathcal{L}(\mathcal{H})$ the Banach space of all the bounded operators on \mathcal{H} and $\{A_n\}_{n \in \mathbb{N}}$ a bounded increasing sequence of non-negative self-adjoint operators in $\mathcal{L}(\mathcal{H})$. Then $\text{s-lim}_{n \rightarrow \infty} A_n$ exists and is a bounded non-negative self-adjoint operator.

(ii) Suppose that $A_1, \dots, A_n, \dots \in \mathcal{L}(L^2(R; \lambda))$ converge to $A \in \mathcal{L}(L^2(R; \lambda))$ strongly, A_n has integral kernel $A_n(x, y)$ for each n and

$$0 \leq A_n(x, y) \uparrow A(x, y) \quad \lambda^{\otimes 2} - \text{a.e. } (x, y) \in R^2.$$

Then A has $A(x, y)$ as its integral kernel.

Proof : For (i), see e.g. [RN].

For (ii), let $f \in L^2(R; \lambda)$. Then $|f| \in L^2(R; \lambda)$ and $(A_n|f|)(x) = \int A_n(x, y)|f(y)|\lambda(dy)$ holds. Taking the limit $n \rightarrow \infty$ (through a subsequence if necessary), we have $(A|f|)(x) = \int A(x, y)|f(y)|\lambda(dy)$ λ -a.e. by strong convergence of the operators and the monotone convergence theorem. The a.e. boundedness of the integral in the righthand side ensures the identity for f instead of $|f|$ by dominated (instead of monotone) convergence theorem. \square

Now we have the following proposition. Here and hereafter, $\|\cdot\|$ and $\|\cdot\|_T$ stand for the operator norm and the trace norm for operators, respectively, and $\|\cdot\|_p$ for the L^p -norm for functions.

Proposition 2.3 (i) Put $K_\Lambda = (K^{1/2}\chi_\Lambda)^*K^{1/2}\chi_\Lambda$ for bounded measurable $\Lambda \subset R$. Then, K_Λ is a bounded non-negative self-adjoint operator and has $K_\Lambda(x, y) \equiv \chi_\Lambda(x)K(x, y)\chi_\Lambda(y)$ as its integral kernel.

$$K_\Lambda = \sum_{k=1}^{\infty} \chi_\Lambda G^k \chi_\Lambda \quad (2.5)$$

holds in the sense of strong convergence of operators.

(ii) For each $k \in \mathbb{N}$, $H_k = \chi_\Lambda G((1 - \chi_\Lambda)G)^{k-1}\chi_\Lambda$ is a bounded non-negative self-adjoint operator having non-negative kernel, denoted by $H_k(x, y)$. $R_\Lambda = \sum_{k=1}^{\infty} H_k$ exists in the strong convergence sense and is the bounded non-negative self-adjoint operator having non-negative kernel $R_\Lambda(x, y) = \sum_{k=1}^{\infty} H_k(x, y)$.

(iii) $R_\Lambda = K_\Lambda(1 + K_\Lambda)^{-1}$, $\|R_\Lambda\| < 1$.

(iv) $(1 + K_\Lambda)^{-1}\chi_\Lambda \geq 0$ a.e. holds, where we regard χ_Λ as a function which belongs to $L^2(R; \lambda)$.

Remark : From (i) of the proposition and the argument above (2.2), it follows that $K_f = \sqrt{1 - e^{-f}}K_\Lambda\sqrt{1 - e^{-f}}$ and its kernel is given by $\sqrt{1 - e^{-f(x)}}K(x, y)\sqrt{1 - e^{-f(y)}}$ for non-negative f satisfying $\text{supp } f \subset \Lambda$.

Proof : (i) Boundedness and self-adjointness of K_Λ are obvious.

Using the spectral decomposition $K = \int_0^\infty \lambda dE_\lambda$, we have

$$G = \int_0^\infty \frac{\lambda}{1 + \lambda} dE_\lambda.$$

Hence,

$$\begin{aligned} \left\| \sum_{k=1}^n \chi_\Lambda G^k \chi_\Lambda \right\| &= \sup_{\|\phi\|_2=1} \sum_{k=1}^n (\chi_\Lambda \phi, G^k \chi_\Lambda \phi) \\ &= \sup_{\|\phi\|_2=1} \int \sum_{k=1}^n \left(\frac{\lambda}{1 + \lambda} \right)^k d(\chi_\Lambda \phi, E_\lambda \chi_\Lambda \phi) \leq \sup_{\|\phi\|_2=1} \int \lambda d(\chi_\Lambda \phi, E_\lambda \chi_\Lambda \phi) \\ &= \sup_{\|\phi\|_2=1} \|K^{1/2}\chi_\Lambda \phi\|_2^2 = \|K_\Lambda\|. \end{aligned}$$

Since $\chi_\Lambda G^k \chi_\Lambda \geq 0$ holds for every $k \in \mathbb{N}$, Lemma 2.2(i) yields the existence of $s\text{-}\lim_{n \rightarrow \infty} \sum_{k=1}^n \chi_\Lambda G^k \chi_\Lambda$. On the other hand, thanks to the monotone convergence theorem, we get (2.5) in the weak sense:

$$\left(\phi, \sum_{k=1}^n \chi_\Lambda G^k \chi_\Lambda \phi \right) = \int \sum_{k=1}^n \left(\frac{\lambda}{1 + \lambda} \right)^k d(\chi_\Lambda \phi, E_\lambda \chi_\Lambda \phi)$$

$$\longrightarrow \int \lambda d(\chi_\Lambda \phi, E_\lambda \chi_\Lambda \phi) = (\phi, K_\Lambda \phi).$$

Thus we have (2.5) in the strong sense.

Lemma 2.2(ii) yields the assertion on the kernel of K_Λ .

(ii) It is obvious that H_k is a bounded non-negative self-adjoint operator for every $k \in \mathbb{N}$. From the non-negativity of the kernel of G^k , we have the non-negativity of the kernel $H_k(x, y)$ and

$$0 \leq H_k(x, y) \leq \chi_\Lambda(x) G^k(x, y) \chi_\Lambda(y).$$

Lemma 2.2(i) and the estimate

$$\begin{aligned} \left\| \sum_{k=1}^n H_k \right\| &= \sup_{\|\phi\|_2=1} \sum_{k=1}^n \int_{R^2} \overline{\phi(x)} H_k(x, y) \phi(y) \lambda^{\otimes 2}(dx dy) \\ &\leq \sup_{\|\phi\|_2=1} \sum_{k=1}^n \int_{R^2} |\phi(x)| \chi_\Lambda(x) G^k(x, y) \chi_\Lambda(y) |\phi(y)| \lambda^{\otimes 2}(dx dy) \leq \left\| \sum_{k=1}^n \chi_\Lambda G^k \chi_\Lambda \right\| \leq \|K_\Lambda\|, \end{aligned}$$

we get the existence of the strong limit R_Λ of $\{\sum_{k=1}^n H_k\}_n$ and its bounded self-adjointness. Lemma 2.2(ii) yields the assertion on the kernel of R_Λ .

(iii) From

$$\begin{aligned} \sum_{k=1}^n H_k - \sum_{k=1}^n \chi_\Lambda G^k \chi_\Lambda &= \sum_{k=1}^n \chi_\Lambda G[(1 - \chi_\Lambda)G]^{k-1} - G^{k-1} \chi_\Lambda \\ &= \sum_{k=2}^n \sum_{l=1}^{k-1} \chi_\Lambda G((1 - \chi_\Lambda)G)^{k-l-1} (-\chi_\Lambda G) G^{l-1} \chi_\Lambda \\ &= - \sum_{l=1}^{n-1} \sum_{k=l+1}^n \chi_\Lambda G((1 - \chi_\Lambda)G)^{k-l-1} \chi_\Lambda \chi_\Lambda G^l \chi_\Lambda = - \sum_{l=1}^{n-1} \sum_{m=1}^{n-l} H_m \chi_\Lambda G^l \chi_\Lambda, \end{aligned}$$

we get the relation

$$\begin{aligned} \sum_{k=1}^n H_k(x, y) - \sum_{k=1}^n \chi_\Lambda(x) G^k(x, y) \chi_\Lambda(y) \\ = - \sum_{l=1}^{n-1} \sum_{m=1}^{n-l} \int_R H_m(x, z) \chi_\Lambda(z) G^l(z, y) \chi_\Lambda(y) \lambda(dz) \quad a.e. \end{aligned}$$

in terms of kernels. Taking the limit $n \rightarrow \infty$, we get

$$R_\Lambda(x, y) - K_\Lambda(x, y) = - \int_R R_\Lambda(x, z) K_\Lambda(z, y) \lambda(dz) \quad \lambda^{\otimes 2} - a.e.(x, y)$$

by the monotone convergence theorem. It implies $R_\Lambda - K_\Lambda = -R_\Lambda K_\Lambda$ and hence $R_\Lambda = K_\Lambda(1 + K_\Lambda)^{-1}$. Since K_Λ is non-negative and bounded, $\|R_\Lambda\| < 1$.

(iv) We may regard G as a contraction operator on $L^\infty(R; \lambda)$ because of (2.1). H_k is also contraction on $L^\infty(R; \lambda)$ for all $k \in \mathbb{N}$. Thus we have

$$\begin{aligned} \sum_{k=1}^n (H_k \chi_\Lambda)(x) &\leq \sum_{k=1}^n (H_k \chi_\Lambda)(x) + (\chi_\Lambda G((1 - \chi_\Lambda)G)^{n-1}(1 - \chi_\Lambda))(x) \\ &\leq \sum_{k=1}^{n-1} (H_k \chi_\Lambda)(x) + (\chi_\Lambda G((1 - \chi_\Lambda)G)^{n-2}(1 - \chi_\Lambda))(x) \leq \cdots \leq (\chi_\Lambda G 1)(x) \leq \chi_\Lambda(x), \end{aligned}$$

where non-negativity of the kernel of G and (2.1) have been used. On the other hand, we get $\sum_{k=1}^n H_k \chi_\Lambda \rightarrow R_\Lambda \chi_\Lambda$ a.e. from (ii) through subsequence if necessary. Hence $(1 + K_\Lambda)^{-1} \chi_\Lambda = \chi_\Lambda - R_\Lambda \chi_\Lambda \geq 0$ a.e. holds. \square

(Proof of Theorem 2.1)

Recall that $K_f = \sqrt{1 - e^{-f}} K_\Lambda \sqrt{1 - e^{-f}}$, for non-negative measurable f and a bounded measurable set $\Lambda \supset \text{supp } f$. Since

$$(1 + (1 - e^{-f})K_\Lambda) \sqrt{1 - e^{-f}} (1 + K_f)^{-1} \sqrt{1 - e^{-f}} = 1 - e^{-f} = 1 - e^{-f} R_\Lambda - e^{-f} (1 + K_\Lambda)^{-1}$$

and

$$1 + (1 - e^{-f})K_\Lambda = (1 - e^{-f} R_\Lambda)(1 + K_\Lambda),$$

we get

$$\begin{aligned} \sqrt{1 - e^{-f}} (1 + K_f)^{-1} \sqrt{1 - e^{-f}} &= (1 + K_\Lambda)^{-1} (1 - e^{-f} R_\Lambda)^{-1} (1 - e^{-f} R_\Lambda - e^{-f} (1 + K_\Lambda)^{-1}) \\ &= (1 + K_\Lambda)^{-1} [1 - (1 - e^{-f} R_\Lambda)^{-1} e^{-f} (1 + K_\Lambda)^{-1}] = (1 + K_\Lambda)^{-1} - (1 + K_\Lambda)^{-1} \sum_{n=0}^{\infty} (e^{-f} R_\Lambda)^n e^{-f} (1 + K_\Lambda)^{-1}. \end{aligned}$$

The Neumann expansion in the last step is valid since $\|e^{-f} R_\Lambda\| \leq \|R_\Lambda\| < 1$. Hence we have

$$\begin{aligned} &-(\sqrt{1 - e^{-f}}, [1 + K_f]^{-1} \sqrt{1 - e^{-f}}) \\ &= -(\chi_\Lambda, (1 + K_\Lambda)^{-1} \chi_\Lambda) + \sum_{l=0}^{\infty} ((1 + K_\Lambda)^{-1} \chi_\Lambda, e^{-f} (R_\Lambda e^{-f})^l (1 + K_\Lambda)^{-1} \chi_\Lambda). \end{aligned}$$

Substituting this identity to the right hand side of (2.3), expanding the exponential and symmetrizing, we get a expression of the form

$$\sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \sigma_{\Lambda^n}(x_1, \dots, x_n) e^{-\sum_{k=1}^n f(x_k)} dx_1 \cdots dx_n \quad (2.6)$$

with a family of symmetric non-negative functions $\{\sigma_{\Lambda^n}\}$ for every $\Lambda \supset \text{supp } f$. For the existence of the measure $\mu_{K, \rho}$ on $Q(R)$, it is enough to show the consistency condition[L]:

$$\sigma_{\Lambda^n}(x_1, \dots, x_n) = \sum_{l=0}^{\infty} \frac{1}{l!} \int_{\Delta^l} \sigma_{(\Lambda \cup \Delta)^{n+l}}(x_1, \dots, x_n, y_1, \dots, y_l) dy_1 \cdots dy_l,$$

where $\Delta \cap \Lambda = \emptyset$. This condition can be derived easily from the facts that the right hand side of (2.3) does not depend on $\Lambda \supset \text{supp } f$ and that for a given Λ , $\{\sigma_{\Lambda^n}\}$ in (2.6) is uniquely determined a.e., since f can be arbitrary non-negative measurable function satisfying $\text{supp } f \subset \Lambda$.

Thus we have proved Theorem 2.1. \square

3 The Thermodynamic Limit

In this section, we follow the arguments and the notations of I §2.2. However, let us review them briefly to make the article self-contained.

Consider $\mathcal{H}_L = L^2(\Lambda_L)$ on $\Lambda_L = [-L/2, L/2]^d \subset \mathbb{R}^d$ for $d > 2$ with the Lebesgue measure on Λ_L . Let Δ_L be the Laplacian under the periodic boundary condition in \mathcal{H}_L . For $k \in \mathbb{Z}^d$, $\varphi_k^{(L)}(x) = L^{-d/2} \exp(i2\pi k \cdot x/L)$ is an eigenfunction of Δ_L , and $\{\varphi_k^{(L)}\}_{k \in \mathbb{Z}^d}$ forms an complete orthonormal system [CONS] of \mathcal{H}_L . The operator $G_L = \exp(\beta \Delta_L)$ has the integral kernel

$$G_L(x, y) = \sum_{k \in \mathbb{Z}^d} e^{-\beta|2\pi k/L|^2} \varphi_k^{(L)}(x) \overline{\varphi_k^{(L)}(y)}, \quad (3.1)$$

for $\beta > 0$. We put $g_k^{(L)} = \exp(-\beta|2\pi k/L|^2)$ which is the eigenvalue of G_L for the eigenfunction $\varphi_k^{(L)}(x)$. We also need the operator $G = \exp(\beta \Delta)$ on $L^2(\mathbb{R}^d)$ and its integral kernel

$$G(x, y) = \int_{\mathbb{R}^d} \frac{dp}{(2\pi)^d} e^{-\beta|p|^2 + ip \cdot (x-y)} = \frac{\exp(-|x-y|^2/4\beta)}{(4\pi\beta)^{d/2}}.$$

Let $f : \mathbb{R}^d \rightarrow [0, \infty)$ be a continuous function of compact support. We will only consider the case where L is so large that Λ_L contains $\text{supp } f$. We regard f as a function on Λ_L naturally.

Let

$$\tilde{G}_L = G_L^{1/2} e^{-f} G_L^{1/2}, \quad (3.2)$$

where e^{-f} represents the operator of multiplication by the function e^{-f} .

Suppose there are N identical particles which obey Bose-Einstein statistics in Λ_L under the periodic boundary condition at inverse temperature β . The basic postulates of quantum mechanics and of statistical mechanics of canonical ensembles yield

$$p_{L,N}^B(x_1, \dots, x_N) = \frac{1}{Z_B N!} \text{per} \{G(x_i, x_j)\}_{i,j=1}^N \quad (3.3)$$

as the probability density distribution of the positions of N particles of the system, where Z_B is the normalization constant and per represents the permanent of matrices. Here, we have set $\hbar^2/2m = 1$. We define the random point field (the probability measure on $Q(\mathbb{R}^d)$) $\mu_{L,N}^B$ induced by the map $\Lambda_L^N \ni (x_1, \dots, x_N) \mapsto \sum_{j=1}^N \delta_{x_j} \in Q(\mathbb{R}^d)$ from the probability measure on Λ_L^N which has the density (3.3). By $E_{L,N}^B$, we denote the expectation with respect to $\mu_{L,N}^B$. The Laplace transform of the point process is given by

$$\begin{aligned} E_{L,N}^B[e^{-\langle f, \xi \rangle}] &= \frac{\int_{\Lambda^N} \exp(-\sum_{j=1}^N f(x_j)) \text{per} \{G_L(x_i, x_j)\}_{i,j=1}^N dx_1 \cdots dx_N}{\int_{\Lambda^N} \text{per} \{G_L(x_i, x_j)\}_{i,j=1}^N dx_1 \cdots dx_N} \\ &= \frac{\int_{\Lambda^N} \text{per} \{\tilde{G}_L(x_i, x_j)\}_{i,j=1}^N dx_1 \cdots dx_N}{\int_{\Lambda^N} \text{per} \{G_L(x_i, x_j)\}_{i,j=1}^N dx_1 \cdots dx_N}. \end{aligned} \quad (3.4)$$

Let us consider the thermodynamic limit, where N and the volume of the box Λ_L tend to infinity in such a way that the densities tend to a positive finite value ρ :

$$L, N \rightarrow \infty, \quad N/L^d \rightarrow \rho > 0. \quad (3.5)$$

In this paper, we concentrate on the high density region

$$\rho > \rho_c = \int_{\mathbb{R}^d} \frac{dp}{(2\pi)^d} \frac{e^{-\beta|p|^2}}{1 - e^{-\beta|p|^2}} \quad (3.6)$$

where the Bose-Einstein condensation takes place.

Theorem 3.1 (i) *The operator $K = G(1 - G)^{-1}$ is a non-negative unbounded self-adjoint operator in $L^2(\mathbb{R}^d)$ and satisfies Condition K in §2. Moreover, K_f defined by (2.2) is a trace class operator.*

(ii) *The finite point fields defined above converge weakly to the random point field whose Laplace transform is given by*

$$\mathbb{E}_\rho^B[e^{-\langle f, \xi \rangle}] = \frac{\exp\left(-(\rho - \rho_c)(\sqrt{1 - e^{-f}}, [1 + K_f]^{-1}\sqrt{1 - e^{-f}})\right)}{\text{Det}[1 + K_f]} \quad (3.7)$$

in the thermodynamic limit (3.5–3.6).

Remark: Thus the resulting point field of the theorem is a convolution of a point field which is an example of those discussed in §2 and a boson process. On the formulation of boson processes, we refer to [ST], where the operator K is assumed to be bounded, however the proof given there is also valid for the present case.

Let us begin the proof with the following lemma, where we use the notation

$$\square_k^{(L)} = \frac{2\pi}{L} \left(k + \left(-\frac{1}{2}, \frac{1}{2} \right]^d \right) \quad \text{for } k \in \mathbb{Z}^d.$$

Lemma 3.2 *For $z \in [0, 1]$, $\nu = 1, 2$ and $L \in [1, \infty)$, let us define functions $a_\nu(\cdot; z), a_\nu^{(L)}(\cdot; z)$ on \mathbb{R}^d by*

$$a_\nu(p; z) = \frac{ze^{-\beta|p|^2}}{(1 - ze^{-\beta|p|^2})^\nu}$$

and

$$a_\nu^{(L)}(p; z) = \begin{cases} 0 & \text{if } p \in \square_0^{(L)} \\ a_\nu(2\pi k/L; z) & \text{if } p \in \square_k^{(L)} \end{cases} \quad \text{for } k \in \mathbb{Z}^d - \{0\}.$$

Then

$$0 \leq a_1^{(L)}(p; z) \leq a_1(2p/(2 + \sqrt{d}); 1) \in L^1(\mathbb{R}^d)$$

and the bounds for large L

$$\frac{L^d}{(2\pi)^d} \int_{\mathbb{R}^d} a_2^{(L)}(p; z) dp \leq \ell(L) \equiv \begin{cases} c_d(L/\sqrt{\beta})^d & \text{if } d > 4 \\ \tilde{c}_4(L/\sqrt{\beta})^4 \log(\tilde{c}L/\sqrt{\beta}) & \text{if } d = 4 \\ c_d(L/\sqrt{\beta})^4 & \text{if } d < 4 \end{cases}$$

hold, where c_d, \tilde{c}_4 and \tilde{c} are positive constants.

Proof: Since a_ν is monotone increasing in z and monotone decreasing as a function of $|p|$, we have

$$\begin{aligned} a_\nu^{(L)}(p; z) &\leq \sup_{L \geq 1} a_\nu^{(L)}(p; 1) \leq \sup\{ a_\nu(q; 1) \mid q \in \mathbb{R}^d, L \geq 1, |q| \geq \frac{2\pi}{L}, |q - p| \leq (2\pi/L)(\sqrt{d}/2) \} \\ &\leq \sup\{ a_\nu(q; 1) \mid q \in \mathbb{R}^d, L \geq 1, |q| \geq \frac{2\pi}{L}, |p| - \pi\sqrt{d}/L \leq |q| \}. \end{aligned}$$

In the case of $|p| \geq (2 + \sqrt{d})\pi$, the last supremum is attained at $L = 1, |q| = |p| - \pi\sqrt{d}$ then $|q| \geq |2p|/(2 + \sqrt{d})$ holds. On the other hand, if $|p| < (2 + \sqrt{d})\pi$, the supremum is attained at $L = (2 + \sqrt{d})\pi/|p|, |q| = 2\pi/L$ and then $|q| = |2p|/(2 + \sqrt{d})$ holds. For both cases, we get the bound $a_\nu^{(L)}(p; z) \leq a_\nu(2p/(2 + \sqrt{d}); 1)$. Since $d > 2$, we get $a_1(2p/(2 + \sqrt{d}); 1) \in L^1(\mathbb{R}^d)$.

Integrating the angular variables, we have

$$\begin{aligned} \frac{L^d}{(2\pi)^d} \int_{\mathbb{R}^d} a_2^{(L)}(p; z) dp &\leq \frac{L^d}{(2\pi)^d} \int_{|p| \geq \pi/L} a_2(2p/(2 + \sqrt{d}); 1) dp \\ &= \left(\frac{L}{2\pi\sqrt{\beta'}} \right)^d S_d \int_{\pi\sqrt{\beta'}/L}^\infty \frac{q^{d-1} e^{-q^2}}{(1 - e^{-q^2})^2} dq = \left(\frac{L}{2\pi\sqrt{\beta'}} \right)^d S_d \mathcal{I}_d, \end{aligned}$$

where $\beta' = 4\beta/(2 + \sqrt{d})^2$. Since $\mathcal{I}_d \leq \int_0^\infty [q^{d-1} e^{-q^2}/(1 - e^{-q^2})^2] dq < \infty$ for $d > 4$; $\mathcal{I}_d \leq \int_{\pi\sqrt{\beta'}/L}^\infty [q^{d-1}/q^4] dq = (4 - d)^{-1}(L/\pi\sqrt{\beta'})^{4-d}$ for $d < 4$ and

$$\mathcal{I}_4 \leq \int_1^\infty \frac{q^{3-1} e^{-q^2}}{(1 - e^{-q^2})^2} dq + \int_{\pi\sqrt{\beta'}/L}^1 \frac{q^3}{q^4} dq = \text{const.} + \log \frac{L}{\pi\sqrt{\beta'}}, \quad (3.8)$$

we get the bounds for $\pi\sqrt{\beta} \leq L$. □

(*Proof of Theorem 3.1(i)*) It is obvious that $K = G(1 - G)^{-1}$ is a unbounded non-negative self-adjoint operator satisfying $G = K(1 + K)^{-1}$. In fact, K is explicitly given by the Fourier transformation:

$$K\phi = \mathcal{F}^{-1}(a_1(\cdot; 1)\mathcal{F}\phi)$$

for

$$\phi \in \text{Dom } K = \{ \psi \in L^2(\mathbb{R}^d) \mid a_1(\cdot; 1)\mathcal{F}\psi \in L^2(\mathbb{R}^d) \}.$$

Condition K(ii) for G are also obvious.

Let us show the locally boundedness of K . For bounded measurable $\Lambda \subset \mathbb{R}^d$,

$$\begin{aligned} \|\sqrt{K}\chi_\Lambda\phi\|_2^2 &= \|\sqrt{a_1(\cdot; 1)}\mathcal{F}(\chi_\Lambda\phi)\|_2^2 \leq \|\sqrt{a_1(\cdot; 1)}\|_2^2 \|\mathcal{F}(\chi_\Lambda\phi)\|_\infty^2 \\ &\leq \|a_1(\cdot; 1)\|_1 \|\chi_\Lambda\phi\|_1^2 \leq (2\pi)^d \rho_c \|\chi_\Lambda\|_2^2 \|\phi\|_2^2. \end{aligned}$$

Thus $K^{1/2}\chi_\Lambda$ is bounded. $K(x, y)$ in (2.4) is given by

$$K(x, y) = \sum_{n=1}^\infty G^n(x, y) = \sum_{n=1}^\infty \int \frac{dp}{(2\pi)^d} e^{-n\beta|p|^2 + ip \cdot (x-y)}$$

$$= \int \frac{dp}{(2\pi)^d} a_1(p; 1) e^{ip \cdot (x-y)},$$

where we have used the dominated convergence theorem. From $a_1 \in L^1(\mathbb{R}^d)$, $K(x, y)$ is continuous. The remark after Proposition 2.3 and the continuity of f yield that the kernel of K_f is continuous. Hence K_f is a trace class operator, because $\|K_f\|_T = \int K_f(x, x) dx = \rho_c \|1 - e^{-f}\|_1 < \infty$. \square

The rest of this section is devoted to the proof of the second part of the theorem. It is obvious from non-negativity of f that $0 \leq \tilde{G}_L \leq G_L$. Let us denote all the eigenvalues of \tilde{G}_L in decreasing order

$$\tilde{g}_0(L) \geq \tilde{g}_1(L) \geq \cdots \geq \tilde{g}_j(L) \geq \cdots.$$

Correspondingly, we relabel the eigenvalues $\{g_k^{(L)}\}_{k \in \mathbb{Z}^d}$ of G_L as

$$g_0(L) = 1 > g_1(L) \geq \cdots \geq g_j(L) \geq \cdots.$$

By the min-max principle, we have

$$g_j(L) \geq \tilde{g}_j(L) \quad (j = 0, 1, 2, \dots).$$

Note that $\varphi_0^{(L)}$ has eigenvalue $g_0(L) = g_0^{(L)} = 1$.

Put

$$D_L = G_L - \tilde{G}_L = G_L^{1/2}(1 - e^{-f})G_L^{1/2}, \quad W_L = G_L^{1/2}\sqrt{1 - e^{-f}},$$

then $D_L = W_L W_L^*$. Note also that

$$\frac{L^d}{(2\pi)^d} \int_{\mathbb{R}^d} a_\nu^{(L)}(p; z) dp = \sum_{k \in \mathbb{Z}^d - \{0\}} \frac{z g_k^{(L)}}{(1 - z g_k^{(L)})^\nu} = \|z Q_0 G_L Q_0 (1 - z Q_0 G_L Q_0)^{-\nu}\|_T. \quad (3.9)$$

Here P_0 is the orthogonal projection on $L^2(\Lambda_L)$ to its one dimensional subspace $\mathbb{C}\varphi_0^{(L)}$ and $Q_0 = 1 - P_0$.

Now we have;

Lemma 3.3 (i) $\|Q_0 G_L Q_0 (1 - Q_0 G_L Q_0)^{-2}\|_T = \sum_{k \neq 0} g_k(L)/(1 - g_k(L))^2 \leq \ell(L)$

In the limit $L \rightarrow \infty$, the following convergences hold.

- (ii) $L^{-d} \text{Tr} G_L \rightarrow \int_{\mathbb{R}^d} e^{-\beta|p|^2} dp / (2\pi)^d = \sqrt{4\pi\beta}^{-d}$, $\|D_L\|_T \rightarrow \|1 - e^{-f}\|_1 / \sqrt{4\pi\beta}^d$
- (iii) $L^{-d} \|Q_0 G_L Q_0 (1 - Q_0 G_L Q_0)^{-1}\|_T = L^{-d} \sum_{k \neq 0} g_k(L)/(1 - g_k(L)) \rightarrow \rho_c < \infty$
- (iv) If $\{z_L\} \subset (0, 1)$ and $z_L \rightarrow 1$, then

$$\sup_{x, y \in \Lambda} |[z_L Q_0 G_L Q_0 (1 - z_L Q_0 G_L Q_0)^{-1}](x, y) - K(x, y)| \rightarrow 0$$

for any fixed bounded measurable set $\Lambda \subset \mathbb{R}^d$.

Proof : (i)–(iii) are immediate consequences of above remarks, Lemma 3.2 and the dominated convergence theorem.

For (iv), put $e(p; x) = e^{ip \cdot x}$ and

$$e^{(L)}(p; x) = e(2\pi k/L; x) \quad \text{if } p \in \square_k^{(L)} \quad \text{for } k \in \mathbb{Z}^d.$$

Then Lemma 3.2 and the dominated convergence theorem also yields

$$\begin{aligned} |[z_L Q_0 G_L Q_0 (1 - z_L Q_0 G_L Q_0)^{-1}](x, y) - K(x, y)]| &\leq \int \frac{dp}{(2\pi)^d} |e^{(L)}(p; x-y) a_1^{(L)}(p; z_L) - e(p; x-y) a_1(p; 1)| \\ &\leq \int \frac{dp}{(2\pi)^d} (|a_1^{(L)}(p; z_L) - a_1(p; 1)| + |e^{(L)}(p; x-y) - e(p; x-y)| a_1(p; 1)) \rightarrow 0. \quad \square \end{aligned}$$

In the followings, we use the notation $B_L = \hat{O}(L^\alpha)$ which means

$$\exists c_1 \geq c_2 > 0 : c_1 L^\alpha \geq B_L \geq c_2 L^\alpha.$$

Lemma 3.4

- (i) For large L , $g_0(L) - \tilde{g}_0(L)$

$$= L^{-d} (\sqrt{1 - e^{-f}}, [1 + W_L^* Q_0 [1 - Q_0 G_L Q_0]^{-1} Q_0 W_L]^{-1} \sqrt{1 - e^{-f}}) (1 + o(1))$$

$$= (\varphi_0^{(L)}, (D_L - D_L Q_0 [1 - Q_0 \tilde{G}_L Q_0]^{-1} Q_0 D_L) \varphi_0^{(L)}) (1 + o(1)).$$
- (ii) $\|1 - e^{-f}\|_1 / L^d = (\varphi_0^{(L)}, D_L \varphi_0^{(L)}) \geq (\varphi_0^{(L)}, D_L Q_0 [\tilde{g}_0(L) - Q_0 \tilde{G}_L Q_0]^{-1} Q_0 D_L \varphi_0^{(L)})$
- (iii) $L^d (g_0(L) - \tilde{g}_0(L)) \in \left[\frac{\|1 - e^{-f}\|_1 (1 + o(1))}{1 + \rho_c \|1 - e^{-f}\|_1}, \|1 - e^{-f}\|_1 \right]$
- (iv) Let $\tilde{\varphi}_0^{(L)}$ be the normalized eigenfunction of \tilde{G}_L for eigenvalue $\tilde{g}_0(L)$ such that $(\tilde{\varphi}_0^{(L)}, \varphi_0^{(L)}) \geq 0$. Put $\tilde{\varphi}_0^{(L)} = a \varphi_0^{(L)} + \varphi'$, $\varphi_0^{(L)} = a' \tilde{\varphi}_0^{(L)} + \tilde{\varphi}'$ ($(\varphi_0^{(L)}, \varphi') = 0$, $(\tilde{\varphi}_0^{(L)}, \tilde{\varphi}') = 0$). Then $a = a'$ and $\|\varphi'\|^2 = \|\tilde{\varphi}'\|^2 = 1 - a^2 = O(L^{-2d} \ell(L))$ hold.

Proof: Here, we suppress the index L in $g_j(L), \tilde{g}_j(L), \varphi_0^{(L)}$ and so on. First notice that $(\varphi_0, D_L \varphi_0) = \|1 - e^{-f}\|_1 / L^d$. From the min-max principle, $d > 2$ and the value of $g_1 = \exp(-\beta|2\pi/L|^2)$, we have

$$g_0 = 1 \geq \tilde{g}_0 \geq (\varphi_0, \tilde{G}_L \varphi_0) = 1 - (\varphi_0, D_L \varphi_0) = 1 - \hat{O}(L^{-d}) > g_1 = 1 - \hat{O}(L^{-2}) \geq \tilde{g}_1 \geq \dots \quad (3.10)$$

for L large enough. Hence the eigenspace of \tilde{G}_L for its largest eigenvalue \tilde{g}_0 is one-dimensional. Let $\tilde{\varphi}_0$ be the normalized eigenfunction for \tilde{g}_0 and put $\tilde{\varphi}_0 = a \varphi_0 + \varphi'$ ($(\varphi_0, \varphi') = 0$), then $\tilde{G}_L \tilde{\varphi}_0 = \tilde{g}_0 \tilde{\varphi}_0$ yields

$$a \tilde{G}_L \varphi_0 + \tilde{G}_L \varphi' = a \tilde{g}_0 \varphi_0 + \tilde{g}_0 \varphi'.$$

Applying P_0 and Q_0 , we have

$$\begin{aligned} a g_0 - a (\varphi_0, D_L \varphi_0) - (\varphi_0, D_L \varphi') &= a \tilde{g}_0 \\ -a Q_0 D_L \varphi_0 + Q_0 \tilde{G}_L \varphi' &= \tilde{g}_0 \varphi'. \end{aligned}$$

Because of $Q_0 \tilde{G}_L Q_0 \leq Q_0 G_L Q_0 \leq g_1 < \tilde{g}_0$, $\tilde{g}_0 - Q_0 \tilde{G}_L Q_0$ is positive, invertible and

$$\varphi' = -a[\tilde{g}_0 - Q_0 \tilde{G}_L Q_0]^{-1} Q_0 D_L \varphi_0, \quad (3.11)$$

$$\begin{aligned} g_0 - \tilde{g}_0 &= (\varphi_0, (D_L - D_L Q_0 [\tilde{g}_0 - Q_0 \tilde{G}_L Q_0]^{-1} Q_0 D_L) \varphi_0) \\ &= (W_L^* \varphi_0, (1 - W_L^* Q_0 [\tilde{g}_0 - Q_0 \tilde{G}_L Q_0]^{-1} Q_0 W_L) W_L^* \varphi_0). \end{aligned} \quad (3.12)$$

For brevity, we put

$$X' = W_L^* Q_0 [\tilde{g}_0 - Q_0 G_L Q_0]^{-1} Q_0 W_L, \quad X = W_L^* Q_0 [1 - Q_0 G_L Q_0]^{-1} Q_0 W_L$$

and

$$\tilde{X} = W_L^* Q_0 [\tilde{g}_0 - Q_0 \tilde{G}_L Q_0]^{-1} Q_0 W_L.$$

Then we have

$$\tilde{X} - X' = -\tilde{X} X',$$

and hence

$$\tilde{X} = X' (1 + X')^{-1} \quad \text{and} \quad 1 - \tilde{X} = (1 + X')^{-1}. \quad (3.13)$$

Together with $W_L^* \varphi_0 = \sqrt{1 - e^{-f}} L^{-d/2}$, we have

$$g_0 - \tilde{g}_0 = L^{-d} (\sqrt{1 - e^{-f}}, (1 + X')^{-1} \sqrt{1 - e^{-f}}) \quad (3.14)$$

from (3.12). Now, we want to replace X' by X in the right hand side. From $1 - \tilde{g}_0 = O(L^{-d})$, $\tilde{g}_0 - g_1 = \hat{O}(L^{-2})$, $\sum_{k \neq 0} g_k / (1 - g_k)^2 \leq \ell(L)$ ((3.10), Lemma 3.3(i)), it follows that

$$\begin{aligned} \|X' - X\| &= (1 - \tilde{g}_0) \|W_L^* Q_0 [\tilde{g}_0 - Q_0 G_L Q_0]^{-1} [1 - Q_0 G_L Q_0]^{-1} Q_0 W_L\| \\ &= (1 - \tilde{g}_0) \sup_{\|\phi\|_2=1} (\phi, W_L^* Q_0 [\tilde{g}_0 - Q_0 G_L Q_0]^{-1} [1 - Q_0 G_L Q_0]^{-1} Q_0 W_L \phi) \\ &\leq (1 - \tilde{g}_0) \sup_{\|\phi\|_2=1} \sum_{k \neq 0} |(\varphi_k, \sqrt{1 - e^{-f}} \phi)|^2 \frac{g_k}{(\tilde{g}_0 - g_k)(1 - g_k)} \\ &\leq (1 - \tilde{g}_0) \sup_{\|\phi\|_2=1} \frac{\|\sqrt{1 - e^{-f}} \phi\|_1^2}{L^d} \frac{1 - g_1}{\tilde{g}_0 - g_1} \sum_{k \neq 0} \frac{g_k}{(1 - g_k)^2} \\ &= \|1 - e^{-f}\|_1 O(L^{-2d} \ell(L)) = o(1). \end{aligned} \quad (3.15)$$

Together with the similar estimate $\|X\| \leq \rho_c \|1 - e^{-f}\|_1 (1 + o(1))$, we have $\|X'\| \leq \rho_c \|1 - e^{-f}\|_1 (1 + o(1))$. Thus (3.14) yields

$$L^d (g_0 - \tilde{g}_0) = (\sqrt{1 - e^{-f}}, (1 + X')^{-1} \sqrt{1 - e^{-f}}) \geq \frac{\|1 - e^{-f}\|_1}{1 + \rho_c \|1 - e^{-f}\|_1} (1 + o(1)),$$

which is the lower bound of (iii). The upper bound of (iii) is obvious. From

$$\begin{aligned} &|(\sqrt{1 - e^{-f}}, (1 + X')^{-1} \sqrt{1 - e^{-f}}) - (\sqrt{1 - e^{-f}}, (1 + X)^{-1} \sqrt{1 - e^{-f}})| \\ &\leq \|\sqrt{1 - e^{-f}}\|_2^2 \|(1 + X)^{-1}\| \|(1 + X')^{-1}\| \|X - X'\| = o(1), \end{aligned}$$

we get the first equality of (i). Replacing X' by X in (3.14) and tracing the argument back to (3.12), we get the second one of (i). (ii) is an immediate consequence of $g_0 \geq \tilde{g}_0$ and (3.12).

(iv) Clearly, $a = (\tilde{\varphi}_0, \varphi_0) = a'$. As for (3.13), we have

$$(\tilde{g}_0 - Q_0 \tilde{G}_L Q_0)^{-1} Q_0 W_L = (\tilde{g}_0 - Q_0 G_L Q_0)^{-1} Q_0 W_L (1 + X')^{-1}. \quad (3.16)$$

This and estimates similar to (3.15) derive the bound

$$\begin{aligned} \|\varphi'\|^2 &= a^2(\varphi_0, D_L Q_0 [\tilde{g}_0 - Q_0 \tilde{G}_L Q_0]^{-2} Q_0 D_L \varphi_0) \\ &\leq a^2 \|W_L^* \varphi_0\|_2^2 \|W_L^* Q_0 [\tilde{g}_0 - Q_0 \tilde{G}_L Q_0]^{-2} Q_0 W_L\| \\ &= a^2 \|W_L^* \varphi_0\|_2^2 \|(1 + X')^{-1} W_L^* Q_0 [\tilde{g}_0 - Q_0 G_L Q_0]^{-2} Q_0 W_L (1 + X')^{-1}\| \\ &= a^2 O(L^{-2d} \ell(L)) \end{aligned}$$

from (3.11). Now the bound for $1 - a^2$ is obvious. \square

As in I, we use the generalized Vere-Jones' formula [V, ST] in the form

$$\frac{1}{N!} \int \text{per}(J(x_i, x_j))_{i,j=1}^N \lambda^{\otimes N}(dx_1 \cdots dx_N) = \oint_{S_r(0)} \frac{dz}{2\pi i z^{N+1}} \text{Det}(1 - zJ)^{-1},$$

where $r > 0$ satisfies $\|rJ\| < 1$. $S_r(\zeta)$ denotes the integration contour defined by the map $\theta \mapsto \zeta + r \exp(i\theta)$, where θ ranges from $-\pi$ to π , $r > 0$ and $\zeta \in \mathbb{C}$. Then we get

$$E_{L,N}^B[e^{-\langle f, \xi \rangle}] = \frac{z_0^N \text{Det}[1 - z_0 G_L] \oint_{S_1(0)} \text{Det}[1 - \tilde{z}_0 \tilde{G}_L (1 - \tilde{z}_0 \tilde{G}_L)^{-1} (\eta - 1)]^{-1} d\eta / 2\pi i \eta^{N+1}}{\tilde{z}_0^N \text{Det}[1 - \tilde{z}_0 \tilde{G}_L] \oint_{S_1(0)} \text{Det}[1 - z_0 G_L (1 - z_0 G_L)^{-1} (\eta - 1)]^{-1} d\eta / 2\pi i \eta^{N+1}}. \quad (3.17)$$

The positive real numbers $z_0 = z_0(L, N)$ and $\tilde{z}_0 = \tilde{z}_0(L, N)$ are chosen as the solutions of the equations

$$\text{Tr}_{\mathcal{H}}[z_0 G_L (1 - z_0 G_L)^{-1}] = \text{Tr}_{\mathcal{H}}[\tilde{z}_0 \tilde{G}_L (1 - \tilde{z}_0 \tilde{G}_L)^{-1}] = N. \quad (3.18)$$

In fact, the following lemma holds.

Lemma 3.5 (i) $z_0 = z_0(L, N) \in (0, 1)$ is uniquely determined by the equation

$$\text{Tr}[z_0 G_L (1 - z_0 G_L)^{-1}] = N. \quad (3.19)$$

(ii) $\tilde{z}_0 = \tilde{z}_0(L, N) \in (0, \tilde{g}_0^{-1}(L))$ is uniquely determined by the equation

$$\text{Tr}[\tilde{z}_0 \tilde{G}_L (1 - \tilde{z}_0 \tilde{G}_L)^{-1}] = N. \quad (3.20)$$

(iii) $0 \leq \tilde{z}_0 - z_0 = O(L^{-d})$

(iv) $1 - z_0 = (1 + o(1))L^{-d}(\rho - \rho_c)^{-1}$

Proof: Let $H(z_0)$ and $\tilde{H}(\tilde{z}_0)$ be the left-hand sides of (3.19) and (3.20), respectively. Since H is monotone increasing continuous function, $H(0) = 0$ and $H(1 - 0) = \infty$, (i) follows. (ii) is similar. The first inequality of (iii) is a consequence of $H(z) \geq \tilde{H}(z)$. Before showing the second inequality, let us make the following remark on the thermodynamic limit (3.5).

(a) If and only if $\rho < \rho_c$, $\{z_0(L, N)\}$ converges to $z = z_* \in (0, 1)$, the unique solution of

$$\rho = \int \frac{dp}{(2\pi)^d} a_1(p; z).$$

(b) If and only if $\rho > \rho_c$, $L^d(1 - z_0) \rightarrow 1/(\rho - \rho_c)$, hence $\lim z_0 = 1$.

(c) If and only if $\rho = \rho_c$, $\lim z_0 = 1$ and $L^d(1 - z_0) \rightarrow +\infty$.

To show (a – c), note that

$$\begin{aligned} & \frac{z_0(L, N)}{L^d(1 - z_0(L, N))} + \int_{\mathbb{R}^d} \frac{dp}{(2\pi)^d} a_1^{(L)}(p; z_0(L, N)) \\ &= \text{Tr}[z_0(L, N)G_L(1 - z_0(L, N)G_L)^{-1}]/L^d = N/L^d \rightarrow \rho. \end{aligned} \quad (3.21)$$

We have that

$$\int_{\mathbb{R}^d} \frac{dp}{(2\pi)^d} a_1^{(L)}(p; z_0) \rightarrow \int_{\mathbb{R}^d} \frac{dp}{(2\pi)^d} a_1(p; z)$$

for $\lim z_0 = z \in [0, 1]$ by the dominated convergence theorem, and that the limit is a strictly increasing function of z . (See Lemma 3.2.) If $\lim z_0 = z_* \in [0, 1)$, the limit of (3.21) tends to

$$\rho = \int_{\mathbb{R}^d} \frac{dp}{(2\pi)^d} a_1(p; z_*) < \rho_c.$$

If $\lim z_0 = 1$, then $\rho = \rho_c + \lim z_0/L^d(1 - z_0) \geq \rho_c$. Now suppose $\{z_0(L, N)\}$ does not converge. Then by taking converging subsequences having different limits, we deduce a contradiction to (3.21). Thus we get the classification (a – c) and (iv).

Now we have the second part of (iii) using Lemma 3.4(iii),

$$z_0 = 1 - \hat{O}(L^{-d}) \leq \tilde{z}_0 < \tilde{g}_0^{-1} = 1 + \hat{O}(L^{-d}). \quad \square$$

In order to understand the subsequent arguments, it is helpful to keep the followings in mind:

$$g_0 = 1, \quad g_1 = 1 - \hat{O}(L^{-2}) \geq Q_0 G_L Q_0 \geq Q_0 \tilde{G}_L Q_0 \quad (\text{see (3.10)})$$

$$\tilde{g}_0 = 1 - \hat{O}(L^{-d}) \quad (\text{Lemma 3.4(iii)})$$

$$z_0 = 1 - \hat{O}(L^{-d}) \quad \tilde{z}_0 = z_0 + O(L^{-d}) \quad (\text{Lemma 3.5(iii, iv)})$$

$$(\varphi_k^{(L)}, D_L \varphi_k^{(L)}) = g_k^{(L)} \|1 - e^{-f}\|_1 / L^d$$

Lemma 3.6

- (i) $P_0[1 - \tilde{z}_0 \tilde{G}_L]^{-1} P_0 = \left(\frac{1}{1 - \tilde{z}_0 \tilde{g}_0} + O(L^{-d} \ell(L)) \right) P_0,$
- (ii) $\|Q_0[1 - \tilde{z}_0 \tilde{G}_L]^{-1}\| = \|[1 - \tilde{z}_0 \tilde{G}_L]^{-1} Q_0\| = O(\sqrt{\ell(L)}),$
 $\|Q_0[1 - z_0 \tilde{G}_L]^{-1}\| = \|[1 - z_0 \tilde{G}_L]^{-1} Q_0\| = O(\sqrt{\ell(L)}),$
- (iii) $\text{Tr}(Q_0[1 - z_0 G_L]^{-1} D_L [1 - z_0 G_L]^{-1} Q_0) = O(L^{-d} \ell(L)).$

Proof: (i) By lemma 3.4(iv), we have

$$\begin{aligned} & |(\varphi_0, (1 - \tilde{z}_0 \tilde{G}_L)^{-1} \varphi_0) - (1 - \tilde{z}_0 \tilde{g}_0)^{-1}| = |(a\tilde{\varphi}_0 + \tilde{\varphi}', (1 - \tilde{z}_0 \tilde{G}_L)^{-1}(a\tilde{\varphi}_0 + \tilde{\varphi}')) - (1 - \tilde{z}_0 \tilde{g}_0)^{-1}| \\ & \leq \frac{1 - a^2}{1 - \tilde{z}_0 \tilde{g}_0} + |(\tilde{\varphi}', (1 - \tilde{z}_0 \tilde{G}_L)^{-1} \tilde{\varphi}')| \leq ((1 - \tilde{z}_0 \tilde{g}_0)^{-1} + (1 - \tilde{z}_0 \tilde{g}_1)^{-1}) O(L^{-2d} \ell(L)) = O(L^{-d} \ell(L)), \end{aligned}$$

where we have used

$$\frac{1}{1 - \tilde{z}_0 \tilde{g}_0} + \frac{1}{1 - \tilde{z}_0 \tilde{g}_1} \leq 2 + \text{Tr}[\tilde{z}_0 \tilde{G}_L (1 - \tilde{z}_0 \tilde{G}_L)^{-1}] = 2 + N = O(L^d),$$

in the last step.

(ii) Note that $Q_0 \tilde{\varphi}_0 = \varphi'$ in the notation of lemma 3.4(iv). Then we get

$$\|Q_0(1 - \tilde{z}_0 \tilde{G}_L)^{-1}\| \leq \frac{\|\varphi'\|}{1 - \tilde{z}_0 \tilde{g}_0} + \frac{1}{1 - \tilde{z}_0 \tilde{g}_1} = O(L^d \sqrt{L^{-2d} \ell(L)}) + O(L^2).$$

The second bound is obtained similarly.

(iii) From the above remark, the left-hand side equals

$$\frac{\|1 - e^{-f}\|_1}{L^d} \sum_{k \neq 0} \frac{g_k}{(1 - z_0 g_k)^2},$$

which yields the righthand side by Lemma 3.3(i). □

We need a finer estimate than Lemma 3.5(iii).

Lemma 3.7

$$(i) \quad \tilde{z}_0 - z_0 = (1 - \tilde{g}_0)(1 + o(1)),$$

$$(ii) \quad 1 - \tilde{z}_0 g'_0 = (1 - \tilde{z}_0 \tilde{g}_0)(1 + o(1)) = (1 - z_0)(1 + o(1)) = \frac{1 + o(1)}{L^d(\rho - \rho_c)},$$

$$\text{where } g'_0 = 1 - (\varphi_0^{(L)}, D_L \varphi_0^{(L)}) + \tilde{z}_0 (\varphi_0^{(L)}, D_L Q_0 [1 - \tilde{z}_0 Q_0 \tilde{G}_L Q_0]^{-1} Q_0 D_L \varphi_0^{(L)}).$$

Proof: (i) Let us begin with

$$\begin{aligned} 0 &= N - N = \text{Tr}[\tilde{z}_0 \tilde{G}_L (1 - \tilde{z}_0 \tilde{G}_L)^{-1} - z_0 G_L (1 - z_0 G_L)^{-1}] \\ &= (\varphi_0, ((1 - \tilde{z}_0 \tilde{G}_L)^{-1} - (1 - z_0 G_L)^{-1}) \varphi_0) + \text{Tr}[Q_0((1 - \tilde{z}_0 \tilde{G}_L)^{-1} - (1 - z_0 \tilde{G}_L)^{-1}) Q_0] \\ &\quad + \text{Tr}[Q_0((1 - z_0 \tilde{G}_L)^{-1} - (1 - z_0 G_L)^{-1}) Q_0]. \end{aligned}$$

The first term of the right hand side equals

$$(1 - \tilde{z}_0 \tilde{g}_0)^{-1} - (1 - z_0 g_0)^{-1} + O(L^{-d} \ell(L)) = \frac{(\tilde{z}_0 - z_0) \tilde{g}_0 - z_0 (g_0 - \tilde{g}_0)}{(1 - \tilde{z}_0 \tilde{g}_0)(1 - z_0 g_0)} + O(L^{-d} \ell(L))$$

by Lemma 3.6(i). On the other hand, the second term has the bound

$$(\tilde{z}_0 - z_0) |\text{Tr}[Q_0(1 - \tilde{z}_0 \tilde{G}_L)^{-1} \tilde{G}_L (1 - z_0 \tilde{G}_L)^{-1} Q_0]|$$

$$\leq \frac{\tilde{z}_0 - z_0}{\tilde{z}_0} \|\tilde{z}_0 \tilde{G}_L (1 - \tilde{z}_0 \tilde{G}_L)^{-1}\|_T \|(1 - z_0 \tilde{G}_L)^{-1} Q_0\| = O(L^{-d} L^d \sqrt{\ell(L)}) = o(L^d)$$

by Lemma 3.5(iii, ii) and Lemma 3.6(ii). The third term can be estimated as

$$\begin{aligned} |\text{Tr}[Q_0((1 - z_0 \tilde{G}_L)^{-1} - (1 - z_0 G_L)^{-1})Q_0]| &= z_0 |\text{Tr}[Q_0(1 - z_0 \tilde{G}_L)^{-1} W_L W_L^* (1 - z_0 G_L)^{-1} Q_0]| \\ &= z_0 |\text{Tr}[Q_0(1 - z_0 G_L)^{-1} W_L (1 + z_0 W_L^* (1 - z_0 G_L)^{-1} W_L)^{-1} W_L^* (1 - z_0 G_L)^{-1} Q_0]| \\ &\leq z_0 \|Q_0(1 - z_0 G_L)^{-1} W_L W_L^* (1 - z_0 G_L)^{-1} Q_0\|_T = O(L^{-d} \ell(L)) = o(L^d), \end{aligned}$$

where we have used a equality similar to (3.16) and Lemma 3.6(iii). Thus we have

$$\frac{z_0(g_0 - \tilde{g}_0) - (\tilde{z}_0 - z_0)\tilde{g}_0}{(1 - \tilde{z}_0\tilde{g}_0)(1 - z_0g_0)} = o(L^d).$$

On the other hand, $(1 - \tilde{z}_0\tilde{g}_0)(1 - z_0g_0) = O(L^{-2d})$ holds. Thus we have

$$z_0(g_0 - \tilde{g}_0) - (\tilde{z}_0 - z_0)\tilde{g}_0 = o(L^{-d}).$$

Note that $g_0 - \tilde{g}_0$ is exactly of order L^{-d} by Lemma 3.4(iii), we get the desired estimate.

(ii) From (3.12), we have

$$\begin{aligned} |\tilde{g}_0 - g'_0| &= |(\varphi_0, D_L Q_0[(\tilde{g}_0 - Q_0 \tilde{G}_L Q_0)^{-1} - (\tilde{z}_0^{-1} - Q_0 \tilde{G}_L Q_0)^{-1}] Q_0 D_L \varphi_0)| \\ &= |(\varphi_0, D_L Q_0(\tilde{g}_0 - Q_0 \tilde{G}_L Q_0)^{-1/2} [(\tilde{z}_0^{-1} - \tilde{g}_0)(\tilde{z}_0^{-1} - Q_0 \tilde{G}_L Q_0)^{-1}] (\tilde{g}_0 - Q_0 \tilde{G}_L Q_0)^{-1/2} Q_0 D_L \varphi_0)| \\ &\leq |\tilde{z}_0^{-1} - \tilde{g}_0| \|(\tilde{z}_0^{-1} - Q_0 \tilde{G}_L Q_0)^{-1}\| (\varphi_0, D_L Q_0(\tilde{g}_0 - Q_0 \tilde{G}_L Q_0)^{-1} Q_0 D_L \varphi_0) \\ &\leq O(L^{-d}) O(L^2) (\varphi_0, D_L \varphi_0) = O(L^{2-2d}) = o(L^{-d}), \end{aligned}$$

where Lemma 3.4(ii) has been used in the last inequality. Hence, we obtain $1 - \tilde{z}_0 g'_0 = 1 - \tilde{z}_0 \tilde{g}_0 + o(L^{-d})$. On the other hand, we have

$$\begin{aligned} 1 - \tilde{z}_0 \tilde{g}_0 &= 1 - z_0 + [\tilde{z}_0(1 - \tilde{g}_0) - (\tilde{z}_0 - z_0)] \\ &= \frac{1 + o(1)}{L^d(\rho - \rho_c)} + o(L^{-d}), \end{aligned}$$

thanks to Lemma 3.5(iv) and (i) above. □

Put

$$p_j^{(N)} = \frac{z_0(L, N) g_j(L)}{1 - z_0(L, N) g_j(L)}, \quad \tilde{p}_j^{(N)} = \frac{\tilde{z}_0(L, N) \tilde{g}_j(L)}{1 - \tilde{z}_0(L, N) \tilde{g}_j(L)},$$

then we have $\sum_{j=0}^{\infty} p_j^{(N)} = \sum_{j=0}^{\infty} \tilde{p}_j^{(N)} = N$ by Lemma 3.5(i, ii),

$$p_0^{(N)} = \hat{O}(L^d), \quad \tilde{p}_0^{(N)} = \hat{O}(L^d), \quad p_0^{(N)} / \tilde{p}_0^{(N)} = 1 + o(1) \quad (3.22)$$

by Lemma 3.7(ii) and

$$p_1^{(N)} = \hat{O}(L^2) \geq p_2^{(N)} \geq \dots, \quad \tilde{p}_1^{(N)} = O(L^2) \geq \tilde{p}_2^{(N)} \geq \dots.$$

Lemma 3.8

$$\oint_{S_1(0)} \frac{1}{\text{Det}[1 - z_0(L, N)G_L(1 - z_0(L, N)G_L)^{-1}(\eta - 1)]} \frac{d\eta}{2\pi i \eta^{N+1}} = \frac{1 + o(1)}{ep_0^{(N)}}$$

$$\oint_{S_1(0)} \frac{1}{\text{Det}[1 - \tilde{z}_0(L, N)\tilde{G}_L(1 - \tilde{z}_0(L, N)\tilde{G}_L)^{-1}(\eta - 1)]} \frac{d\eta}{2\pi i \eta^{N+1}} = \frac{1 + o(1)}{e\tilde{p}_0^{(N)}}$$

Proof : Set $R^{(N)} = \tilde{R}^{(N)} = L^{(d-2)/2}$. Since $\sum_{j=1}^{\infty} p_j^{(N)}(1 + p_j^{(N)}) = \text{Tr}[z_0 Q_0 G_L Q_0 (1 - z_0 Q_0 G_L Q_0)^{-2}] \leq \sum_{j=1}^{\infty} g_j / (1 - g_j)^2$, we get

$$\frac{R^{(N)2} \sum_{j=1}^{\infty} p_j^{(N)}(1 + p_j^{(N)})}{p_0^{(N)2}} \rightarrow 0$$

by $p_0^{(N)} = \hat{O}(L^d)$ and Lemma 3.3(i). Then Lemma A.2 yields

$$\text{the l.h.s. of the 1st eq.} = \oint_{S_1(0)} \frac{1}{\prod_{j=0}^{\infty} (1 - p_j^{(N)}(\eta - 1))} \frac{d\eta}{2\pi i \eta^{N+1}} = \frac{1 + o(1)}{ep_0^{(N)}}.$$

For the second equality, we notice that $\tilde{p}_j^{(N)} \leq (1 + o(1))p_j^{(N)}$ holds for all $j = 1, 2, \dots$, because of $z_0, \tilde{z}_0 = 1 + O(L^{-d})$ and $\tilde{g}_j^{(N)} \leq g_j^{(N)} \leq 1 - \hat{O}(L^{-2})$. Together with (3.22), we have

$$\frac{\tilde{R}^{(N)2} \sum_{j=1}^{\infty} \tilde{p}_j^{(N)}(1 + \tilde{p}_j^{(N)})}{\tilde{p}_0^{(N)2}} \leq (1 + o(1)) \frac{R^{(N)2} \sum_{j=1}^{\infty} p_j^{(N)}(1 + p_j^{(N)})}{p_0^{(N)2}} \rightarrow 0.$$

Thus the second equality also follows from Lemma A.2. \square

Now we have

$$E_{L,N}^B[e^{-\langle f, \xi \rangle}] = \frac{z_0^N}{\tilde{z}_0^N} \frac{\text{Det}[1 - z_0 G_L]}{\text{Det}[1 - \tilde{z}_0 \tilde{G}_L]} (1 + o(1))$$

from (3.17), (3.22) and the above lemma. Since P_0, Q_0 and G_L commute, $\text{Det}[1 - z_0 G_L] = (1 - z_0)\text{Det}[1 - z_0 Q_0 G_L Q_0]$. We use the Feshbach formula to get

$$\begin{aligned} \text{Det}[1 - \tilde{z}_0 \tilde{G}_L] &= \text{Det} \begin{pmatrix} P_0 - \tilde{z}_0 P_0 \tilde{G}_L P_0 & -\tilde{z}_0 P_0 \tilde{G}_L Q_0 \\ -\tilde{z}_0 Q_0 \tilde{G}_L P_0 & Q_0 - \tilde{z}_0 Q_0 \tilde{G}_L Q_0 \end{pmatrix} \\ &= \text{Det}_{Q_0 \mathcal{H}_L}[Q_0 - \tilde{z}_0 Q_0 \tilde{G}_L Q_0] \\ &\quad \times \text{Det}_{P_0 \mathcal{H}_L}[P_0 - \tilde{z}_0 P_0 \tilde{G}_L P_0 - \tilde{z}_0 P_0 \tilde{G}_L Q_0 (Q_0 - \tilde{z}_0 Q_0 \tilde{G}_L Q_0)^{-1} \tilde{z}_0 Q_0 \tilde{G}_L P_0] \\ &= \text{Det}[1 - \tilde{z}_0 Q_0 \tilde{G}_L Q_0] \\ &\quad \times (1 - \tilde{z}_0 [1 - (\varphi_0^{(L)}, D_L \varphi_0^{(L)}) + \tilde{z}_0 (\varphi_0^{(L)}, D_L Q_0 [1 - \tilde{z}_0 Q_0 \tilde{G}_L Q_0]^{-1} Q_0 D_L \varphi_0^{(L)})]) \end{aligned}$$

where Det is the Fredholm determinant for operators on \mathcal{H}_L and $\text{Det}_{Q_0 \mathcal{H}_L}$ for operators on the subspace $Q_0 \mathcal{H}_L$ etc. Now from Lemma 3.7(ii) and Lemma 3.5(iii, iv), we get

$$E_{L,N}^B[e^{-\langle f, \xi \rangle}] = \frac{z_0^N}{\tilde{z}_0^N} \frac{(1 - z_0)\text{Det}[1 - z_0 Q_0 G_L Q_0]}{(1 - \tilde{z}_0 g'_0)\text{Det}[1 - \tilde{z}_0 Q_0 \tilde{G}_L Q_0]} (1 + o(1))$$

$$\begin{aligned}
&= \frac{z_0^N}{\tilde{z}_0^N} \frac{\text{Det}[1 - z_0 Q_0 G_L Q_0]}{\text{Det}[1 - \tilde{z}_0 Q_0 \tilde{G}_L Q_0]} (1 + o(1)) \\
&= \exp\left(-\frac{\tilde{z}_0 - z_0}{z_0} N + o(1)\right) \frac{\text{Det}[1 - z_0 Q_0 G_L Q_0]}{\text{Det}[1 - Q_0 G_L Q_0]} \frac{\text{Det}[1 - Q_0 G_L Q_0]}{\text{Det}[1 - Q_0 \tilde{G}_L Q_0]} \frac{\text{Det}[1 - Q_0 \tilde{G}_L Q_0]}{\text{Det}[1 - \tilde{z}_0 Q_0 \tilde{G}_L Q_0]}. \quad (3.23)
\end{aligned}$$

Lemma 3.9

$$\begin{aligned}
\text{(i)} \quad & \frac{\text{Det}[1 - z_0 Q_0 G_L Q_0]}{\text{Det}[1 - Q_0 G_L Q_0]} = \exp\left(\frac{1 - z_0}{z_0} (N - p_0^{(N)}) + o(1)\right) \\
\text{(ii)} \quad & \frac{\text{Det}[1 - \tilde{z}_0 Q_0 \tilde{G}_L Q_0]}{\text{Det}[1 - Q_0 \tilde{G}_L Q_0]} = \exp\left(\frac{1 - \tilde{z}_0}{\tilde{z}_0} (N - \tilde{p}_0^{(N)}) + o(1)\right) \\
\text{(iii)} \quad & \frac{\text{Det}[1 - Q_0 \tilde{G}_L Q_0]}{\text{Det}[1 - Q_0 G_L Q_0]} = \text{Det}[1 + K_f] (1 + o(1))
\end{aligned}$$

Proof : Put $h(z) = -\log \text{Det}(1 - z Q_0 G_L Q_0) = -\sum_{j=1}^{\infty} \log(1 - z g_j)$, and we have

$$\log \frac{\text{Det}[1 - z_0 Q_0 G_L Q_0]}{\text{Det}[1 - Q_0 G_L Q_0]} = h(1) - h(z_0) = h'(z_0)(1 - z_0) + \frac{1}{2} h''(\bar{z}_0)(1 - z_0)^2,$$

where $\bar{z}_0 \in (z_0, 1)$. Hence we get (i) by

$$h'(z_0) = \sum_{j=1}^{\infty} \frac{g_j}{1 - z_0 g_j} = \frac{N - p_0}{z_0}$$

and

$$h''(\bar{z}_0)(1 - z_0)^2 = \sum_{j=1}^{\infty} \frac{g_j^2 (1 - z_0)^2}{(1 - \bar{z}_0 g_j)^2} \leq \sum_{j=1}^{\infty} \frac{g_j (1 - z_0)^2}{(1 - g_j)^2} = O(L^{-2d} \ell(L)) = o(1),$$

where Lemma 3.3(i) has been used. Similar argument and $\sum_{j=1}^{\infty} \tilde{p}_j (1 + \tilde{p}_j) \leq (1 + o(1)) \sum_{j=1}^{\infty} p_j (1 + p_j)$ yield (ii).

(iii) Thanks to the product and cyclic properties of the Fredholm determinant, we have

$$\begin{aligned}
\frac{\text{Det}[1 - Q_0 \tilde{G}_L Q_0]}{\text{Det}[1 - Q_0 G_L Q_0]} &= \text{Det}[1 + Q_0 (G_L - \tilde{G}_L) Q_0 (1 - Q_0 G_L Q_0)^{-1}] = \text{Det}[1 + W_L^* Q_0 (1 - Q_0 G_L Q_0)^{-1} Q_0 W_L] \\
&= \text{Det}[1 + \sqrt{1 - e^{-f}} Q_0 G_L Q_0 (1 - Q_0 G_L Q_0)^{-1} \sqrt{1 - e^{-f}}].
\end{aligned}$$

Note that $L^2(\Lambda_L)$ can be identified with an closed subspace of $L^2(\mathbb{R}^d)$ naturally. By this identification, we regard G_L and $\sqrt{1 - e^{-f}}$ as operators on $L^2(\mathbb{R}^d)$. Now for (iii), it is enough to prove

$$A_L = \sqrt{1 - e^{-f}} Q_0 G_L Q_0 (1 - Q_0 G_L Q_0)^{-1} \sqrt{1 - e^{-f}} \longrightarrow K_f$$

in the trace norm. In the following, we show $A_L \rightarrow K_f$ strongly and $\|A_L\|_T \rightarrow \|K_f\|_T$. Then the Grüm's convergence theorem [Si] yields the above.

For $\psi, \phi \in L^2(\mathbb{R}^d)$, we have

$$\begin{aligned}
|(\psi, (A_L - K_f)\phi)| &= \left| \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy \overline{\psi(x)} \sqrt{1 - e^{-f(x)}} \phi(y) \sqrt{1 - e^{-f(y)}} \right. \\
&\quad \left. ([Q_0 G_L Q_0 (1 - Q_0 G_L Q_0)^{-1}](x, y) - K(x, y)) \right| \\
&\leq (|\psi|, \sqrt{1 - e^{-f}})(\sqrt{1 - e^{-f}}, |\phi|) \sup_{x, y \in \text{supp } f} |[Q_0 G_L Q_0 (1 - Q_0 G_L Q_0)^{-1}](x, y) - K(x, y)| \quad (3.24) \\
&\leq \|\psi\|_2 \|\phi\|_2 \|\sqrt{1 - e^{-f}}\|_2^2 \sup_{x, y \in \text{supp } f} |[Q_0 G_L Q_0 (1 - Q_0 G_L Q_0)^{-1}](x, y) - K(x, y)|,
\end{aligned}$$

which tends to 0, by Lemma 3.3(iv). Thus the strong (in fact the norm) convergence has been proved. For the convergence of the trace norm, we use Lemma 3.3(iv) again and positive self-adjointness of operators A_L and K_f to get

$$\begin{aligned}
\|A_L\|_T - \|K_f\|_T &= \text{Tr}[A_L - K_f] \\
&= \int_{\mathbb{R}^d} dx (1 - e^{-f(x)}) ([Q_0 G_L Q_0 (1 - Q_0 G_L Q_0)^{-1}](x, x) - K(x, x)) \rightarrow 0.
\end{aligned}$$

□

By Lemma 3.5 (iii, iv), we have

$$-\frac{\tilde{z}_0 - z_0}{z_0} N + \frac{1 - z_0}{z_0} N - \frac{1 - \tilde{z}_0}{\tilde{z}_0} N = \frac{(\tilde{z}_0 - z_0)(1 - \tilde{z}_0)}{z_0 \tilde{z}_0} N = O\left(\frac{1}{N}\right).$$

Applying Lemma 3.4(i) and Lemma 3.7(ii) to the righthand side of

$$-\frac{1 - z_0}{z_0} p_0 + \frac{1 - \tilde{z}_0}{\tilde{z}_0} \tilde{p}_0 = -\frac{1 - \tilde{g}_0}{1 - \tilde{z}_0 \tilde{g}_0},$$

we get the formula

$$\begin{aligned}
&\mathbb{E}_{L,N}^B[e^{-\langle f, \xi \rangle}] = \\
&\frac{\exp\left(-(\rho - \rho_c)(\sqrt{1 - e^{-f}}, [1 + W_L^* Q_0 (1 - Q_0 G_L Q_0)^{-1} Q_0 W_L]^{-1} \sqrt{1 - e^{-f}}) + o(1)\right)}{\text{Det}[1 + K_f]}. \quad (3.25)
\end{aligned}$$

From the convergence $W_L^* Q_0 (1 - Q_0 G_L Q_0)^{-1} Q_0 W_L = A_L \rightarrow K_f$ in the thermodynamic limit, we have proved the theorem.

A Complex integrals

Lemma A.1 For $0 \leq x \leq 1$ and $p \geq 0$ satisfying $0 \leq px < 1$, we have

$$1 \geq (1+x)^p(1-px) \geq \exp\left(-\frac{p(1+p)(1+px^2)}{2(1-px)^2}x^2\right).$$

Proof: Put $f(x) = \log(1+x)^p(1-px)$, then

$$f'(x) = \frac{p}{1+x} - \frac{p}{1-px}, \quad f''(x) = -\frac{p(1+p)(1+px^2)}{(1+x)^2(1-px)^2}$$

hold. So we have $f(0) = 0$, $f'(0) = 0$ and $0 \geq f''(\theta x) \geq -p(1+p)(1+px^2)/(1-px)^2$ for $\theta \in (0, 1)$, which imply the result. \square

Lemma A.2 Let the collection of numbers $\{p_j^{(N)}\}_{j,N}$ satisfies

$$p_0^{(N)} > p_1^{(N)} \geq p_2^{(N)} \geq \dots \geq p_j^{(N)} \geq \dots \geq 0, \quad \sum_{j=0}^{\infty} p_j^{(N)} = N.$$

Suppose that there exist a sequence $\{R^{(N)}\}_{N \in \mathbb{N}}$ and $c \in (0, 1)$ such that

$$1 < R^{(N)} < cp_0^{(N)} \left(1 \wedge \frac{1}{p_1^{(N)}}\right), \quad \lim_{N \rightarrow \infty} p_0^{(N)} / R^{(N)} e^{c'R^{(N)}} = 0$$

and

$$\lim_{N \rightarrow \infty} \frac{R^{(N)2} \sum_{j=1}^{\infty} p_j^{(N)} (1 + p_j^{(N)})}{p_0^{(N)2}} = 0, \quad \text{where } c' = \frac{\log(1+c)}{c}.$$

Then

$$\lim_{N \rightarrow \infty} p_0^{(N)} \oint_{S_1(0)} \frac{d\eta}{2\pi i} \frac{1}{\eta^{N+1} \prod_{j=0}^{\infty} (1 - p_j^{(N)}(\eta - 1))} = \frac{1}{e}$$

holds.

Proof: We omit the superscript (N) here.

Note that $p_0 \rightarrow \infty$ and $R \rightarrow \infty$ as $N \rightarrow \infty$.

By the preceding lemma,

$$1 \geq \prod_{j=1}^{\infty} \left[\left(1 + \frac{R}{p_0}\right)^{p_j} \left(1 - \frac{Rp_j}{p_0}\right) \right] \geq \exp\left(-\sum_{j=1}^{\infty} \frac{p_j(1+p_j)}{2} \left(1 + \frac{R^2 p_j}{p_0^2}\right) \frac{R^2}{p_0^2} \left(1 - \frac{Rp_j}{p_0}\right)^{-2}\right).$$

So the assumption on R implies

$$\prod_{j=1}^{\infty} \left[\left(1 + \frac{R}{p_0}\right)^{p_j} \left(1 - \frac{Rp_j}{p_0}\right) \right] \xrightarrow{N \rightarrow \infty} 1.$$

Similarly, we have

$$\prod_{j=1}^{\infty} \left[\left(1 + \frac{1}{p_0} \right)^{p_j} \left(1 - \frac{p_j}{p_0} \right) \right] \xrightarrow{N \rightarrow \infty} 1.$$

Now let us deform the integration contour of the complex variable η to two parts

$$\oint_{S_1(0)} = - \oint_{S_{(R-1)/p_0}(1+1/p_0)} + \oint_{S_{1+R/p_0}(0)} = I_1 + I_2.$$

I_1 is obtained by the residue at $\eta = 1 + 1/p_0$:

$$\begin{aligned} I_1 &= -p_0 \left[\left(1 + \frac{1}{p_0} \right)^{N+1} (-p_0) \prod_{j=1}^{\infty} \left(1 - \frac{p_j}{p_0} \right) \right]^{-1} \\ &= \left(1 + \frac{1}{p_0} \right)^{-p_0-1} \prod_{j=1}^{\infty} \left[\left(1 + \frac{1}{p_0} \right)^{p_j} \left(1 - \frac{p_j}{p_0} \right) \right]^{-1} \xrightarrow{N \rightarrow \infty} e^{-1}. \end{aligned}$$

I_2 can be estimated as

$$\begin{aligned} |I_2| &\leq p_0 \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \prod_{j=0}^{\infty} \left[\left(1 + \frac{R}{p_0} \right)^{p_j} \left| 1 - p_j \left(\left(1 + \frac{R}{p_0} \right) e^{i\theta} - 1 \right) \right| \right]^{-1} \\ &\leq p_0 \left(1 + \frac{R}{p_0} \right)^{-p_0} |1 - R|^{-1} \left[\prod_{j=1}^{\infty} \left(1 + \frac{R}{p_0} \right)^{p_j} \left(1 - \frac{Rp_j}{p_0} \right) \right]^{-1} \xrightarrow{N \rightarrow \infty} 0, \end{aligned}$$

since $(1 + R/p_0)^{p_0} \geq (1 + c)^{R/c} = e^{c'R}$ and the assumption. □

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